Section 8-4, Mathematics 104

Matrices and Determinants, Cramer's Rule

A **matrix** is an array of numbers. The plural of matrix is matrices.

Example:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

This is a 2 x 2 matrix. It has four numbers, each in a specific location

Matrices can have any number of rows or columns,

Example:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} or \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

Matrices are a mathematical object like a number.
Like a number, two matrices can be added if they have the same number of rows and columns:

Example:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} 1+5 & 2+6 \\ 3+7 & 4+8 \end{bmatrix} = \begin{bmatrix} 6 & 8 \\ 10 & 12 \end{bmatrix}$$

You can also multiply a matrix by a number

$$5 \cdot \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 5\cdot 1 & 5\cdot 2 \\ 5\cdot 3 & 5\cdot 4 \end{bmatrix} = \begin{bmatrix} 5 & 10 \\ 15 & 20 \end{bmatrix}$$

You can multiply two matrices, however it is more complicated than it might seem. You do not just multiply each number.

Solving 2 equations in two unknowns using matrixes

To solve two equations in two unknowns we first put the equations in standard form

$$\begin{aligned} -3x + y &= -4 \\ -x + 2y &= -3 \end{aligned}$$

We then transfer the coefficients and constants to a 3 x 2 matrix

$$\begin{bmatrix} -3 & 1 & -4 \\ -1 & 2 & -3 \end{bmatrix}$$

Our goal is to transform this matrix to one that looks like this:

$$\begin{bmatrix} 1 & 0 & a \\ 0 & 1 & b \end{bmatrix}$$

Why do we do this? Because if our transformation uses the rules of manipulating equations properly we end up with the equations:

$$\begin{aligned} x + 0y &= a \\ 0x + y &= b \end{aligned}$$

in which case our solution is (a,b).

So how can we do this?

There are three steps we can take.

1) Switch two rows. This just switches two equations.

2) Multiply the numbers in a row by a constant. This is just multiplying both sides of an equation by the same number.

3) Multiply a row by a constant and add or subtract one row from another. This is what we do all the time when we use elimination.

Here's how this would work.

$$\begin{bmatrix} -3 & 1 & -4 \\ -1 & 2 & -3 \end{bmatrix}$$

First multiply the 2nd row by -4 and add it to the first row, putting a 1 in the upper left

$$\begin{bmatrix} 1 & -7 & 8 \\ -1 & 2 & -3 \end{bmatrix}$$

Now add the first row to the second, putting a 0 in the lower left

$$\begin{bmatrix} 1 & -7 & 8 \\ 0 & -5 & 5 \end{bmatrix}$$

Multiply the 2nd row by -1/5 putting a 1 in the lower right

$$\begin{bmatrix} 1 & -7 & 8 \\ 0 & 1 & -1 \end{bmatrix}$$

Now multiply the lower row by 7 and add it to the top row

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$

We now see that our solution is $(1,-1)$

Everyone should try this:

$$\begin{aligned} 4x+3y &= 11 \\ 4x-y &= 23 \end{aligned}$$

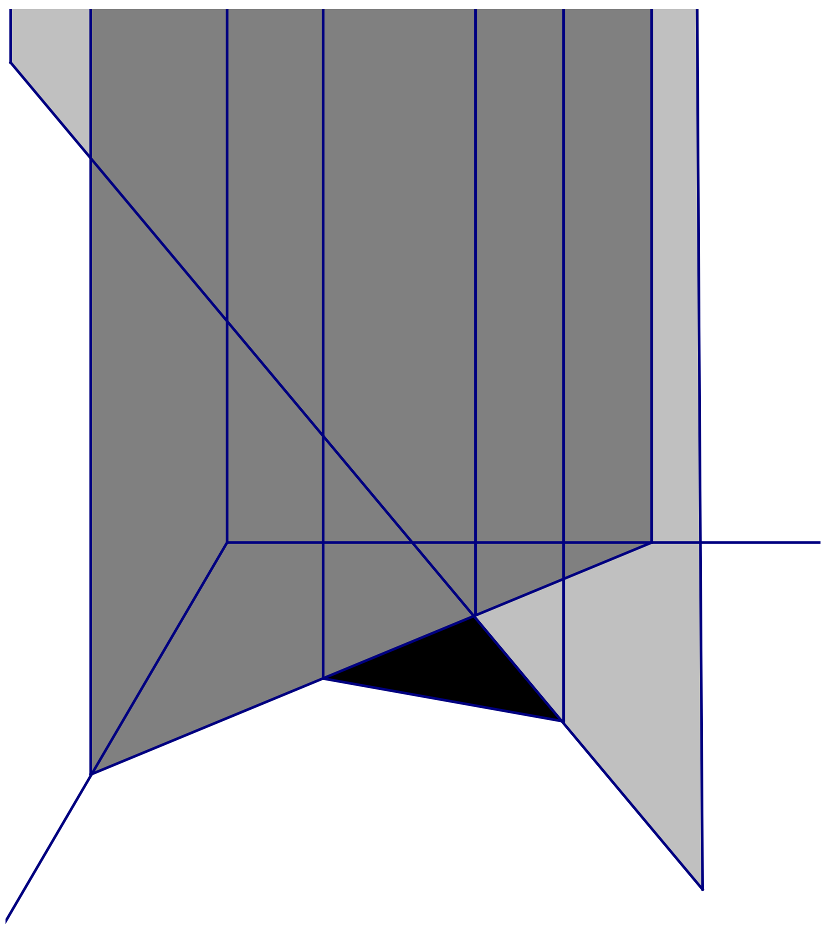
Systems of equations without solutions.

A system of 2 equations in 2 unknowns can have no solutions when the two lines are parallel.

The situation with 3 equations in 3 unknowns is even worse. An equation in three unknowns is a plane in 3D space.

Two or all three of the three planes might be parallel.

Even if they are not parallel, there might not be one point where they intersect.



Solving systems of equations using Cramer's rule.

To understand Cramer's rule, we first have to define the determinant of a matrix.

For a 2 x 2 matrix this is easy to understand.

$$\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix}$$

The determinant of a matrix is a number which we calculate as

D = 1 x 4 - 3 x 2 = -2

Note we multiply along the diagonals and then subtract.
Try these:

$$\begin{vmatrix} 2 & -1 \\ 1 & -2 \end{vmatrix}$$

$$\begin{vmatrix} 1 & 0 \\ 0 & 2 \end{vmatrix}$$

To see what this has to do with solving a system of equations, we first solve an equation in the abstract.

$$ax + by = c$$
$$dx + ey = f$$
$$aex + bey = ce$$
$$dbx + bey = bf$$
$$x\left(ae - db\right) = ec - bf$$
$$x = \frac{ec - bf}{ae - db} = \frac{\begin{vmatrix} c & b \\ f & e \end{vmatrix}}{\begin{vmatrix} a & b \\ d & e \end{vmatrix}}$$

The same way we get

$$y = \frac{\begin{vmatrix} a & c \\ d & f \end{vmatrix}}{\begin{vmatrix} a & b \\ d & e \end{vmatrix}}$$

The denominator is both cases is the determinant of $\begin{bmatrix} a & b \\ d & e \end{bmatrix}$ the four coefficients of x and y.

The numerator is the determinant of the same matrix but for x= the x row is replaced by $\begin{bmatrix} c \\ f \end{bmatrix}$

For y= we replace the y row by $\begin{bmatrix} c \\ f \end{bmatrix}$

Let's do an example:

$$x + 2y = 5$$
$$-x + y = 1$$

$$x = \frac{\begin{vmatrix} 5 & 2 \\ 1 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & 2 \\ -1 & 1 \end{vmatrix}} = \frac{5-2}{1+2} = 1 \qquad y = \frac{\begin{vmatrix} 1 & 5 \\ -1 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & 2 \\ -1 & 1 \end{vmatrix}} = \frac{1+5}{1+2} = 2$$

Try this one

$$3x + 2y = -1$$
$$4x + 5y = -11$$

Like Gaussian elimination, this works for 3 equations in 3 unknowns and 4 equations in 4 unknowns. The complication is that above 3 x 3 a determinant is hard to calculate.